M435

Assignment Booklet I

Contents **Cut-off date**

2	TMA M435 01 (Unit 1 of MTS and GT)	31 March 1994
7	TMA M435 02 (Units 2 and 3 of MTS and GT)	2 June 1994

Please send your answers for each tutor-marked assignment (TMA), together with an appropriately completed assignment form (PT3), to reach your tutor on or before the cut-off date given above.

The marks allocated to each part of a question are indicated in the margin.

Although many of the questions on these assignments require numerical answers, the questions invariably carry *method marks*.

In phrasing the questions we have used the words listed below, each of which should be interpreted as indicated.

prove, show, explain	Clear reasoning and explanation for all steps are called for.
determine, find, devise, calculate	An indication of the method used and all working in arriving at an answer should be given.
deduce	Clear explanation of how one result follows from another is required.
solve	Working must be shown. A numerical answer alone is not sufficient.
evaluate, give, write down	Answer suffices. No explanation need be given.
hence	No marks will be awarded for any alternative method.

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1.1

Questions 1 to 4 are on Metric and Topological Spaces.
Questions 5 to 9 are on Geometric Topology.

Metric and Topological Spaces

Question 1 – 13 marks

For each positive integer n, let

$$A_n = \left\{(x_1, x_2) \in \mathbf{R}^2 : x_2 > \frac{1}{n}x_1\right\}.$$

(i) Sketch the subsets A_1, A_2 of $\mathbf{R}^2$. [2]

(ii) Prove that if $x = (x_1, x_2)$ and $x_1 > 0$, $x_2 > 0$, then there exists a positive integer m such that $x \in A_m$. [4]

(iii) Prove that

$$\bigcup_{n=1}^{\infty} A_n = B \cup C,$$

where

$$B = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \leq 0, x_2 > x_1\}$$

and

$$C = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 > 0, x_2 > 0\}.$$ [7]

Question 2 – 12 marks

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be the function defined by

$$f(x_1, x_2) = |x_1|\,|x_2 - 1|, \quad (x_1, x_2) \in \mathbf{R}^2.$$

(i) Write down the following sets:

(a) $f(\mathbf{R}^2)$;

(b) $f([-1, 1] \times [2, 3])$;

(c) $f^{-1}(\{0\})$. [6]

(ii) Sketch the following subsets of $\mathbf{R}^2$:

(a) $f^{-1}(\{1\})$;

(b) $f^{-1}([0, 1])$. [6]

Question 3 – 12 marks

Let $f : \mathbf{R} \to \mathbf{R}^2$ be the function given by

$$f(x) = (2 + |x|,\ 2 - |x|), \quad x \in \mathbf{R}.$$

(i) Sketch the subset $f(\mathbf{R})$ of $\mathbf{R}^2$. [2]

(ii) Show that

$$f^{-1}(\{(3, 1)\}) = \{-1, 1\}.$$ [2]

(iii) Determine the following subsets of $\mathbf{R}$:

(a) $f^{-1}([2, 5] \times [1, 2])$;

(b) $f^{-1}([2, 5] \times [-2, 2])$. [8]

Question 4 – 13 marks

For each positive integer n, let

$$A_n = \left[0, 1 + \frac{1}{n}\right], \quad B_n = \left[-\frac{1}{n}, \frac{1}{n}\right].$$

(i) Sketch the subsets $A_1 \times B_1$, $A_2 \times B_2$ of $\mathbf{R}^2$. [2]

(ii) Show that if $x = (x_1, x_2)$ and either $x_1 > 1$ or $x_2 \neq 0$, then there is some positive integer n such that $x \notin A_n \times B_n$. [6]

(iii) Determine the subset

$$\bigcap_{n=1}^{\infty} (A_n \times B_n)$$

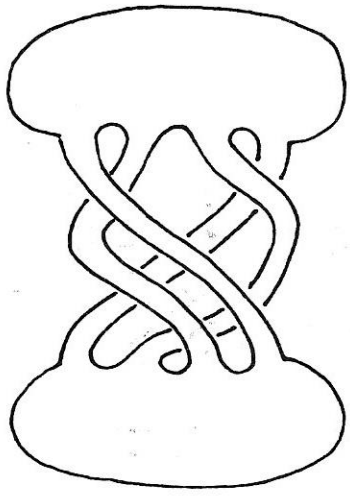
of $\mathbf{R}^2$. [5]

Geometric Topology

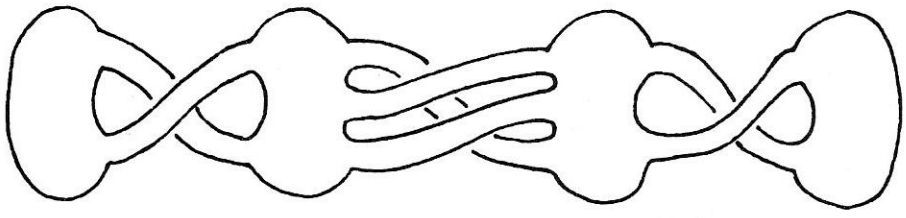
Question 5 – 5 marks

(i) Write down the number of boundary curves of the following two surfaces (made of thin paper). In each case, give a sketch to illustrate your answer.

(a)



(b)



[4]

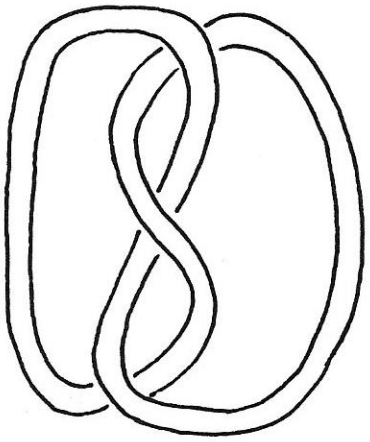
(ii) Explain why the two surfaces above are not homeomorphic. [1]

Question 6 – 5 marks

Before attempting this question, you might like to look at the first Exercise Booklet for the Geometric Topology half of the course, where the terms 'sketch' and 'simpler' are defined.

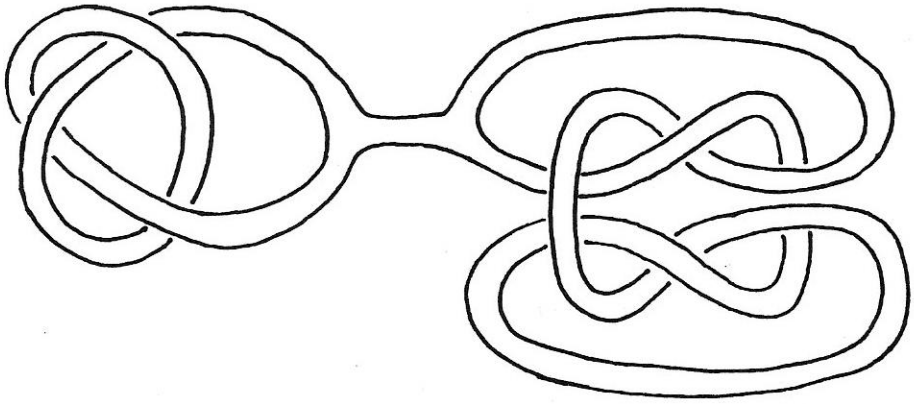
Sketch simpler versions of the following surfaces. In each case explain, preferably with a sketch, how you arrived at the simplification. Both the surfaces are made of thin paper.

(i)



[2]

(ii)

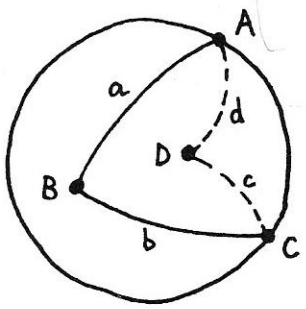


[3]

Question 7 – 15 marks

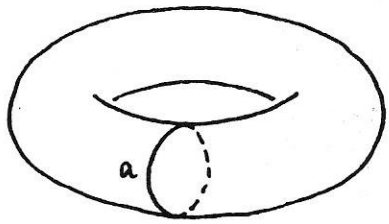
For each of the following surfaces, add any vertices or edges you feel are necessary to obtain a subdivision. State which, if any, of the rules on page 27 of Geometric Topology *Unit 1* were not satisfied by the original figure. Clearly label each vertex and edge that you add.

(i)



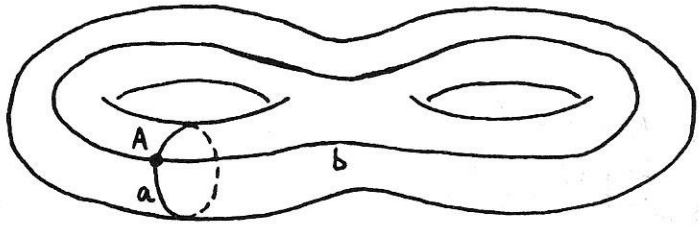
A sphere with four vertices and edges of an inscribed tetrahedron

(ii)



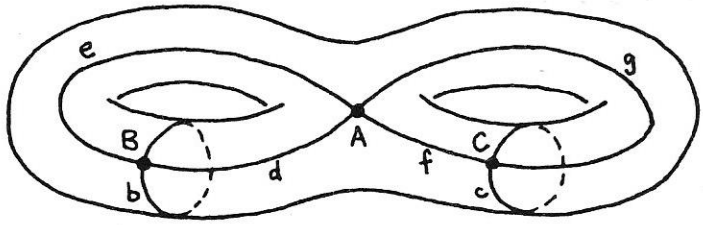
A torus

(iii)



A double torus

(iv)



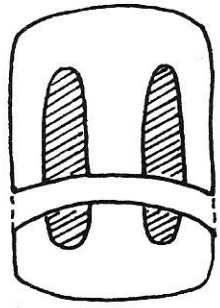
A double torus

[*15*]

Question 8 – 10 marks

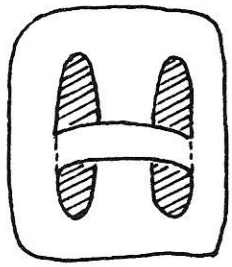
Mark a subdivision on each of the following surfaces, which are made out of thin paper. Holes have been shaded.

(i)



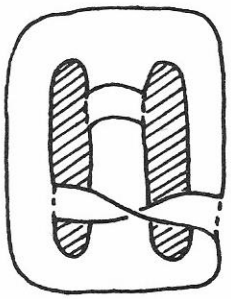
[*2*]

(ii)



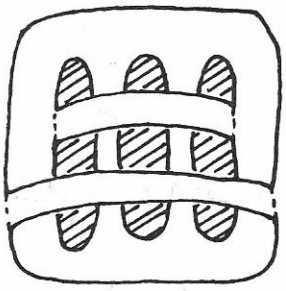
[*3*]

(iii)



[2]

(iv)



[3]

Question 9 – 15 marks

(i) For each of the subdivisions obtained in Question 8, write down the corresponding number of vertices (N_0), edges (N_1) and faces (N_2). [4]

(ii) For each of the above subdivisions, calculate the corresponding value of the Euler characteristic χ. [3]

(iii) For each surface in Question 8, calculate the number, β, of its boundary components. [3]

(iv) The Euler characteristic of an n-holed torus with β boundary components is $2 - 2n - \beta$. Use this fact together with your answers to parts (ii) and (iii) to describe each surface in Question 8 as a suitable torus. In each case write down the corresponding values of n and β. You may assume that the surfaces are orientable. [5]

Questions 1 to 4 are on Metric and Topological Spaces.
Questions 5 to 8 are on Geometric Topology.

Metric and Topological Spaces

Question 1 (Unit 2) – 12 marks

(i) Let A be a non-empty set of real numbers which is bounded above by α. Prove that $\sup A = \alpha$ if and only if for each $\varepsilon > 0$, we can find $a \in A$ such that

$$\alpha - \varepsilon < a \leq \alpha.$$ [4]

(ii) Let A, B be non-empty sets of real numbers which are bounded above, and let C be the set of all numbers of the form $a + b$ where $a \in A$, $b \in B$.

(a) Show that C is bounded above. [2]

(b) Prove that if $\sup A = \alpha$ and $\sup B = \beta$, then $\sup C = \alpha + \beta$. [6]

Question 2 (Unit 2) – 13 marks

(i) Let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = 1 + 3x^2, \quad x \in \mathbf{R}.$$

Show that given $\varepsilon > 0$,

$$|f(x) - f(1)| < \varepsilon$$

whenever $|x - 1| < \delta$, if we take $\delta = \text{Min}\left\{1, \frac{\varepsilon}{9}\right\}$. [5]

(ii) For $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(x) = 3x^2 + x + 5, \quad x \in \mathbf{R},$$

given $\varepsilon > 0$, find a $\delta > 0$ such that

$$|g(x) - g(1)| < \varepsilon$$

whenever $|x - 1| < \delta$. [6]

(iii) Deduce that the function $h : \mathbf{R} \to \mathbf{R}$ given by

$$h(x) = \frac{1 + 3x^2}{3x^2 + x + 5}, \quad x \in \mathbf{R},$$

is continuous at 1. [2]

(You may use any result from the unit, provided that a reference is given.)

Question 3 (Unit 3) – 5 marks

For each of the following subsets of $\mathbf{R}^2$, state whether it is OPEN or NOT OPEN in $\mathbf{R}^2$ with the Euclidean metric.

$$H_1 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 + x_2 \neq 0\}$$
$$H_2 = H_1 \cup \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0\}$$
$$H_3 = H_1 \cup \{(x_1, x_2) \in \mathbf{R}^2 : x_2 < 0\}$$
$$H_4 = H_1 \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0\}$$
$$H_5 = H_1 \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_2 < 0\}$$ [5]

Question 4 (Unit 3) – 20 marks

Let $d : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ be defined as follows: if $x = (x_1, x_2) \in \mathbf{R}^2$ and $y = (y_1, y_2) \in \mathbf{R}^2$ and $x \neq y$, then

$$d(x, y) = \text{Max}\{|x_1|, |x_2|\} + \text{Max}\{|y_1|, |y_2|\},$$

whilst $d(x, y) = 0$ if $x = y$.

(i) Prove that d is a metric on $\mathbf{R}^2$. [8]

(ii) In the metric space $M = \{\mathbf{R}^2, d\}$, determine the open balls:

(a) $B_1((0, 0))$;

(b) $B_1((1, 1))$. [6]

(iii) Prove that the set

$$\{(x_1, x_2) \in \mathbf{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$$

is open in the metric space M. [6]

Geometric Topology

Question 5 (Unit 2) – 13 marks

This question concerns surfaces with Euler characteristic $\chi = 2$, 1 or 0. Each surface has a regular subdivision consisting of k-gons meeting j at a vertex. Let V be the number of vertices, E the number of edges, and F the number of faces, in such a regular subdivision.

(i) Write down a relationship connecting k, F and E. [1]

(ii) Write down a relationship connecting j, V and E. [1]

(iii) Hence write down an equation connecting j, k, F and χ. [1]

For the rest of the question, assume that $j = k$ and $k \geq 3$.

(iv) Express k as a function of χ and F. [2]

(v) Hence determine possible values of V, E, F and k when

(a) $\chi = 2$,

(b) $\chi = 1$,

(c) $\chi = 0$. [5]

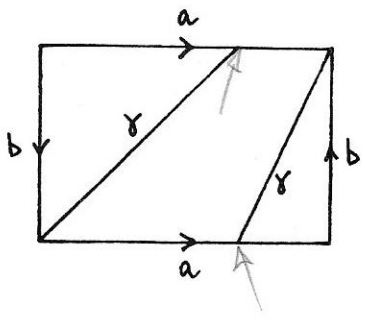
(vi) For each of your answers to (a) and (c) above, give a sketch of a surface or a topological polygon displaying the regular subdivision. [3]

Question 6 (Unit 2) – 12 marks

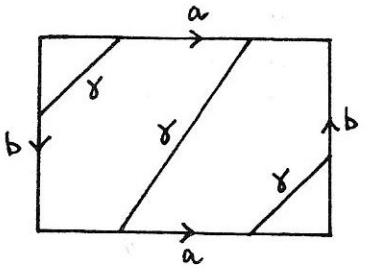
In order to answer part (i) of this question, you will find it helpful to refer to Section 2.4 of the first Exercise Booklet for Geometric Topology.

(i) Find a neighbourhood for each of the following curves γ, and mark the neighbourhood on a sketch. In each case state, giving reasons, whether γ is orientation-preserving or orientation-reversing.

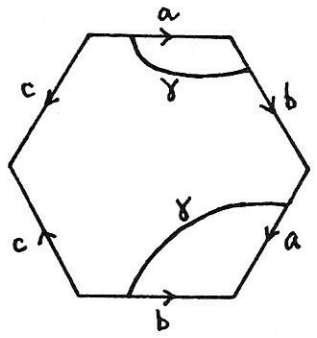
(a)



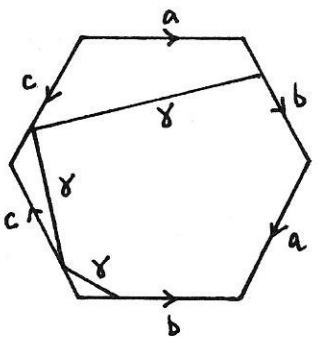
(b)



(c)



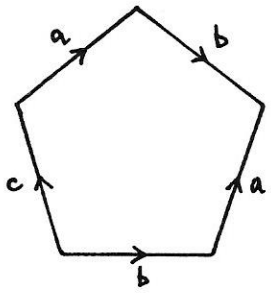
(d)



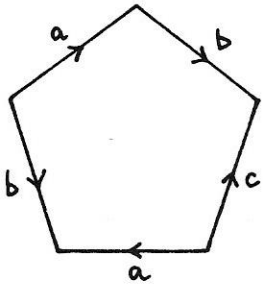
[8]

(ii) Two of the surfaces represented by the following diagrams are non-orientable. Identify them by sketching a Möbius band in each.

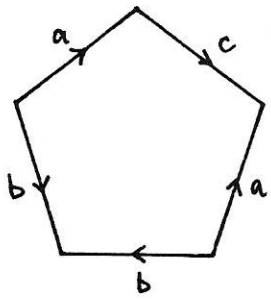
(a)



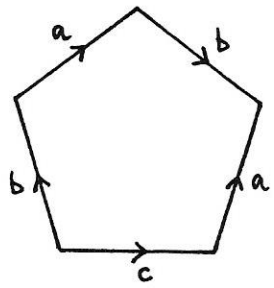
(b)



(c)



(d)



[4]

Question 7 (Unit 3) – 9 marks

Establish the following combinatorial equivalences.

(i) $aba^{-1}cdb^{-1}d^{-1}c^{-1} = 1 \longrightarrow aba^{-1}b^{-1} = 1$ [3]

(ii) $abacb^{-1}c = 1 \longrightarrow aabb = 1$ [3]

(iii) $abcda^{-1}ede^{-1} = 1 \longrightarrow aabbc = 1$ [3]

Question 8 (Unit 3) – 16 marks

(i) For each of the following edge equations, draw a polygon with edge identifications having the given equation as its edge equation.

(a) $aba^{-1}b^{-1}cc^{-1} = 1$

(b) $abca^{-1}b^{-1}c^{-1} = 1$

(c) $abab^{-1}cc^{-1} = 1$

(d) $abc^{-1}a^{-1}bc^{-1} = 1$ [2]

(ii) Two of the above surfaces are tori. Identify which two these are, and for each one:

(a) show that its original equation is combinatorially equivalent to $xyx^{-1}y^{-1} = 1$;

(b) give a sketch (or succession of sketches) to show how joining identified edges in pairs produces a torus. [8]

(iii) Two of the above surfaces are Klein bottles. Identify which two these are, and for each one:

(a) show that its original equation is combinatorially equivalent to $xyxy^{-1} = 1$;

(b) mark a Möbius band on your sketch in part (i). [6]

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M435

Assignment Booklet II

Contents **Cut-off date**

2	TMA M435 03 (Units 4 and 5 of MTS and GT)	4 August 1994
5	TMA M435 04 (Units 6 and 7 of MTS and GT)	29 September 1994

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1.1

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Metric and Topological Spaces

Question 1 (Unit 4) – 14 marks

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$.

(i) Show that

$$\mathscr{T} = \{\varnothing, \{1, 2\}, \{3, 4\}, A\}$$

is a topology on A. [2]

(ii) Explain why

$$\mathscr{B} = \{\varnothing, \{a, b\}, \{a, c\}, B\}$$

is not a topology on B. [2]

(iii) Find the coarsest topology $\mathscr{V}$ on B such that $\mathscr{B} \subset \mathscr{V}$. [3]

(iv) Show that $f : A \to B$ given by

$$f(1) = f(2) = a, \quad f(3) = f(4) = c$$

is $(\mathscr{T}, \mathscr{V})$-continuous. [3]

(v) Show that $g : B \to A$ given by

$$g(a) = 1, \quad g(b) = 2, \quad g(c) = 3, \quad g(d) = 4$$

is not $(\mathscr{V}, \mathscr{T})$-continuous. [2]

(vi) State whether or not $h : A \to B$ given by

$$h(1) = h(2) = a, \quad h(3) = c, \quad h(4) = d$$

is $(\mathscr{T}, \mathscr{V})$-continuous, and justify your answer. [2]

Question 2 (Unit 4) – 11 marks

Let $\mathbf{N}$ and $\mathbf{O}$ denote the sets of positive integers and odd positive integers respectively. For each positive integer n, let

$$V_n = \mathbf{O} \cup \{k \in \mathbf{N} : k \geq n\}.$$

(i) Prove that the collection $\mathscr{V}$ of subsets of $\mathbf{N}$ consisting of $\varnothing$ and the sets V_n, $n = 1, 2, \ldots$, is a topology on $\mathbf{N}$. [5]

(ii) Let $H = \{3, 4, 5, 6\}$.

(a) Show that $\{3, 5\}$ and $\{3, 5, 6\}$ are open sets of the topology induced by $\mathscr{V}$ on H. [3]

(b) Determine the topology induced by $\mathscr{V}$ on H. [3]

Question 3 (Unit 5) – 10 marks

Let $\mathscr{V}$ be the topology on the set $\mathbf{N}$ of positive integers introduced in Question 2 above.

(i) Prove that $\{2,4\}$ is closed in $\{\mathbf{N},\mathscr{V}\}$. [2]

(ii) Find all closed sets in $\{\mathbf{N},\mathscr{V}\}$. [4]

(iii) Find the closure of $\{2,3,5\}$ in $\{\mathbf{N},\mathscr{V}\}$. [2]

(iv) State whether or not $\{\mathbf{N},\mathscr{V}\}$ is a Hausdorff space, and justify your answer. [2]

Question 4 (Unit 5) – 15 marks

Let A denote the open interval $(0,1)$ and let $\mathscr{T}$ be the topology on A which consists of $\varnothing$, A and the intervals $\left(\frac{1}{3n}, \frac{n}{n+1}\right)$ for all positive integers n.

[You need *not* verify that $\mathscr{T}$ is a topology on A.]

(i) Write down the closed sets of $\{A,\mathscr{T}\}$. [2]

(ii) Let $H = \{\frac{1}{3}, \frac{2}{3}\}$.

(a) Show that if $x \in A$, then x is a limit point of H if and only if $x < \frac{1}{3}$ or $x \geq \frac{1}{2}$.

(b) Find the closure of H in $\{A,\mathscr{T}\}$. [9]

(iii) Write down the closure of $K = \{\frac{1}{2}, \frac{2}{3}\}$ in $\{A,\mathscr{T}\}$, and hence determine the set of limit points of K. [4]

Geometric Topology

Question 5 (Unit 4) – 11 marks

(i) Using the technique of inserting vertices, determine χ, the Euler characteristic, and β, the number of boundary curves, for each of the surfaces described by the systems of edge equations below.

(a) $acbfb^{-1}ca^{-1}ef^{-1}d = 1$.

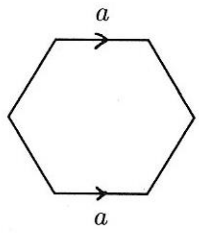
(b) $abdb^{-1} = 1;\ ac^{-1}fe^{-1} = 1;\ def = 1$. [4]

(ii) For each of the surfaces in part (i), determine whether or not the surface is orientable, and write it in connected sum form. [3]

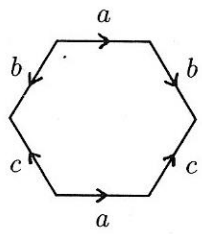
(iii) For each of the systems of equations in part (i), reduce it to canonical form. Justify each step of your working. You may quote Lemmas 1, 2 and 3 of **B** (pages 66–68) and the results of Steps 3 and 4 of the reduction procedure (**B** pages 70–73). [4]

Question 6. (Unit 4) – 14 marks

This question concerns surfaces without boundary which can be obtained from a topological hexagon with a fixed pair of opposite edges identified as shown.



One such surface is the following.



(i) Classify the surface obtained from the second hexagon above. Show your working, and write your answer in connected sum form. [2]

(ii) In the first topological hexagon, the unlabelled edges are to be identified in pairs to produce a surface without boundary. Show that whatever pairing of the edges is chosen, the Euler characteristic χ of the resulting surface cannot be less than -1. [2]

[Hint: What are the possible numbers of vertices after identification of edges?]

(iii) For each different surface which can be constructed as in part (ii) above:

(a) write down the surface as a connected sum;

(b) sketch a hexagon with its edges labelled to correspond to the surface (remember that the edges marked a are assumed to have been marked in advance). [10]

[Hint: There are five different surfaces that can be obtained.]

Question 7 (Unit 5) – 11 marks

Consider the complex locus

$$w^2 - 2wz - \tfrac{1}{4}(5z^3 - 11z + 2) = 0.$$

(i) Determine all infinite points on the locus. [5]

(ii) Find the three (finite) values of z which have only one solution (z, w) on the locus. [Hint: $z = 1$ is one value.] [2]

(iii) Write down the general solutions w_1, w_2 for the two points lying above the point z on the locus, in terms of z and the three solutions t_1, t_2 and t_3 which you found in part (ii). Arguing as in Frames 18–20 of the tape section, show that the three points corresponding to t_1, t_2 and t_3 are branch points of the locus. [4]

Question 8 (Unit 5) – 14 marks

Each of the following conics is non-degenerate.

(i) $z^2 - 2zw + w^2 - 3z + 1 = 0$

(ii) $2z^2 - 2zw + w^2 - 3z + 1 = 0$

For each of these conics, do the following.

(a) Working from first principles, locate all finite branch points, if any. [5]

(b) Determine all infinite points on the conic. [5]

(c) Using the fact that a conic has either two branch points or none, write down which points (if any) are branch points and which points (if any) are pinch points. [2]

(d) Use the main theorem (*Unit 5*, page 30) to confirm your answers. [2]

TMA M435 04 **Cut-off date** 29 September 1994

Questions 1 to 5 are on Metric and Topological Spaces.
Questions 6 to 9 are on Geometric Topology.

Metric and Topological Spaces

Question 1 (Unit 6) – 6 marks

Prove that the subspace

$$H = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 \geq |x_1|\}$$

of $\mathbf{R}^2$ with the usual topology is not compact, by writing down an appropriate open cover of H and showing that it does not have a finite subcover. [6]

Question 2 (Unit 6) – 12 marks

Let $A = [0,1] \times [0,1]$ and let $\mathscr{T}$ be the topology on A which has all sets of the form $[a,1] \times [0,b]$ for $(a,b) \in A$ as a basis. [You need not verify that this collection of sets is a synthetic basis.]

(i) Show that $\{A, \mathscr{T}\}$ is a compact space. [5]

(ii) Let $B = \{(x_1, x_2) \in A : x_1 = \frac{1}{2}\}$.

(a) Show that B is not closed in $\{A, \mathscr{T}\}$.

(b) Show that the subspace B of $\{A, \mathscr{T}\}$ is compact. [5]

(iii) Deduce that $\{A, \mathscr{T}\}$ is not a Hausdorff space. [2]

Question 3 (Unit 6) – 10 marks

Let

$$A = \{(x_1, x_2) \in \mathbf{R}^2 : |x_1| \leq 2,\ x_1^2 + x_2^2 \geq 1\}$$

and let B and C be the subsets of A defined as follows:

$$B = \{(x_1, x_2) \in A : |x_2| < 3\}$$

and $$C = \{(x_1, x_2) \in A : |x_2| \leq 2\}.$$

For each of the subsets A, B and C of $\mathbf{R}^2$ with its usual topology, state whether or not it is: (a) bounded; (b) closed; (c) compact. For each subset, justify your answer to (c). [10]

Question 4 (Unit 7) – 12 marks

Let $A = \{a, b, c, d, e\}$ and let $\mathscr{T}$ be the coarsest topology on A which contains the subsets $\{a\}, \{a, b, c\}$ and $\{b, c, d, e\}$.

(i) Show that the topological space $\{A, \mathscr{T}\}$ is not connected. [5]

(ii) Show that $\{c, d, e\}$ with the induced topology as a subspace of $\{A, \mathscr{T}\}$ is connected. [5]

(iii) Find the connected components of $\{A, \mathscr{T}\}$. [2]

Question 5 (Unit 7) – 10 marks

Let

$$A = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 \leq 1\},$$
$$B = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 = 2\}$$
and $$C = \{(x_1, x_2) \in \mathbf{R}^2 : |x_1| \geq 1,\ x_1^2 + x_2^2 = 2\}.$$

Show that no two of the subspaces A, B and C of $\mathbf{R}^2$ with the usual topology are homeomorphic. [10]

Geometric Topology

Question 6 (Unit 6) – 12 marks

(i) For each of the following coverings, give a diagram to indicate that the covering can occur.

(a) $5T^2 \xrightarrow{2} 3T^2$

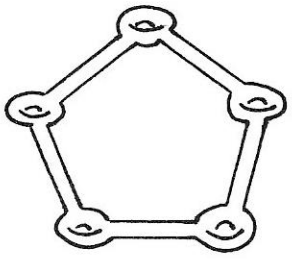
(b) $7T^2 \xrightarrow{2} 4T^2$

(c) $4T^2 \xrightarrow{3} 2T^2$

(d) $6T^2 \xrightarrow{5} 2T^2$ [4]

(ii) Let $kT^2 \xrightarrow{n} pT^2$ be an unbranched n-fold covering of a torus with p holes (i.e. of genus p) by a torus with k holes, where $p > 1$. Obtain an equation connecting n, k and p. [2]

(iii) Let S be an orientable surface without boundary. Let S' be the surface $nS\#T^2$. S' can be realized as n copies of S joined by n tubes so that each copy of S lies at the vertex of a regular n-gon. This configuration is illustrated below for S a torus and $n = 5$.



(To see that in general S' equals $nS\#T^2$, remove one handle from S'. The surface so obtained is a connected sum of n copies of S, which is nS. Putting the handle back, we see that $S' = nS\#T^2$.)

(a) S' is an n-fold unbranched cover of what surface? Write your answer as a connected sum of copies of S and T^2. [3]

(b) By taking S to be $(p-1)T^2$, show that every unbranched n-fold covering of a torus with p holes ($p > 1$) by a torus with k holes which is permitted by the equation obtained in part (ii) above can be realized. [3]

Question 7 (Unit 6) – 13 marks

A 5-sheeted covering $\tilde{S}$ of $S = 2\mathbb{RP}^2 \# D^2$ with edge equation $aabbcdc^{-1} = 1$ is represented by the following system of edge equations.

$$a_1a_2b_1b_3c_1d_1c_2^{-1} = 1$$
$$a_2a_1b_2b_1c_2d_2c_4^{-1} = 1$$
$$a_3a_4b_3b_2c_3d_3c_1^{-1} = 1$$
$$a_4a_5b_4b_5c_4d_4c_3^{-1} = 1$$
$$a_5a_3b_5b_4c_5d_5c_5^{-1} = 1$$

(i) Determine which points of S, if any, are covered by branch points, and in each such case determine the branch deficit. [4]

(ii) Determine the Euler characteristic $\chi(\tilde{S})$ of $\tilde{S}$. [2]

(iii) Determine the number $\beta(\tilde{S})$ of boundary curves of $\tilde{S}$. [2]

(iv) Is $\tilde{S}$ orientable? Justify your answer. [3]

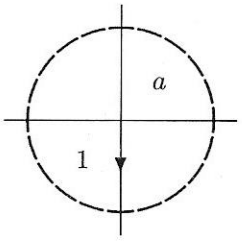
(v) Write down $\tilde{S}$ as a connected sum of copies of T^2, D^2 and/or $\mathbb{RP}^2$. [2]

Question 8 (Unit 7) – 12 marks

(i) Each of the following diagrams represents part of a single closed curve with a sense of direction, from which some of the arrows showing the direction, and some of the winding numbers, have been omitted. All points, except those where the curve crosses itself, are passed through only once.

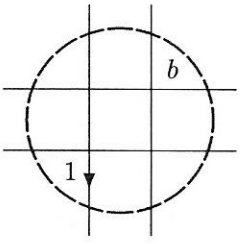
In each case, answer the accompanying question and give an appropriate set of additions to a copy of the diagram to support your answer. Your completed figure need not show a complete single closed curve, but enough arrows and winding numbers to support your claim. You may not draw additional parts of the curve inside the dashed region. All intersections are true crossings, not versions of this where two parts of the curve merely touch.

(a)



What are the possible values of a? [2]

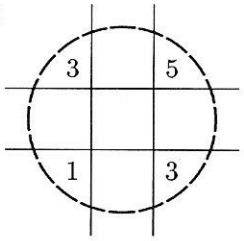
(b)



What are the possible values of b? [2]

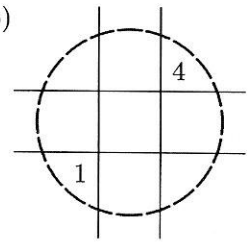
(ii) With the same interpretation of diagrams as before, and with the same restrictions concerning the dashed region, state which of the following can be part of a single closed curve with the given winding numbers, and which cannot. Accompany your answer with a sketch if such a curve is possible or a brief explanation why such a curve is impossible, as appropriate.

(a)



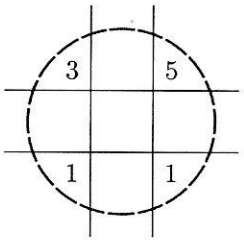
[2]

(b)



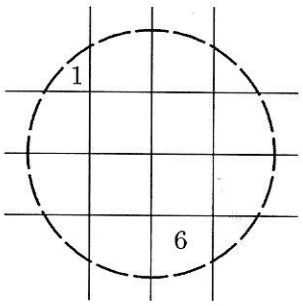
[2]

(c)



[2]

(d)



[2]

Question 9 (Unit 7) – 13 marks

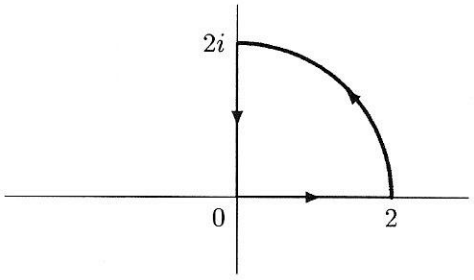
(i) Show that all zeros of $F(z) = z^6 + z + 1$ lie inside the circle $C_2 = \{z : |z| = 2\}$. [2]

(ii) Show that $F(z)$ has no zeros inside the circle $C_{1/2} = \{z : |z| = \frac{1}{2}\}$. [2]

(iii) Show that $F(z)$ has no real zeros. [1]

(iv) Show that $F(z)$ has no purely imaginary zeros. [1]

(v) By considering the image under F of the contour below, find out how many zeros of $F(z)$ have both real and imaginary parts positive.



(The contour consists of the real axis from 0 to 2, followed by the quarter-circle of radius 2, centre the origin, from 2 to $2i$, followed by the imaginary axis from $2i$ to 0.) [5]

(vi) Hence determine the number of zeros of $F(z)$ in each quadrant. [2]

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